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High-frequency excitation of quantum systems with adiabatic nonlinearity

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Abstract. The excitation by a high-frequency field of multi-level quantum systems with a slowly varying density of states is investigated. This class of systems includes hydrogen-like atoms, surface electrons in metals, charge bubbles in liquid helium and other bound systems. It is found that the excitation takes place through a ladder of sharp quasi-resonances, whose shape is universal, namely independent of the driving-field parameters and of the details of the system. The amplitudes of these peaks satisfy a system-dependent tight-binding equation in energy space. Two classes of examples are considered in detail: for a particle in a positive power-law potential well, the amplitudes exhibit a local crossover in energy between a regime of exponential decay and an asymptotic power-law tail, which depends on the field parameters. For a negative power-law potential well, exponential localization, similar to the Anderson localization in a finite lattice, is found. The localization length depends on the field parameters as well as on the specific power of the potential well. The two classes contain, as special cases, the ‘bubble’ model and the one-dimensional hydrogen atom; previous results are confirmed for these cases, and new results are presented.

1. Introduction

The excitation of multi-level quantum systems by external fields, is a fundamental problem in quantum mechanics. Any interaction between matter and electromagnetic fields is described, on the simplest level, by such a model. Given the system at some initial condition, one would like to characterize its energy absorption from the external field as a function of time. An exact solution to this problem, which takes into account a large number of quantum levels, is known only for a linear system, for which the Hamiltonian is quadratic in momentum and coordinate [1]. Standard methods for nonlinear systems are limited: perturbation theory, leading for example to the Fermi golden rule, holds only for very short times, and a two-level approximation (associated with the Rabi width) is invalid in the high-frequency regime, where many levels are nearly resonant.

In some cases, simple quantum systems can be described by one degree of freedom [2, 3]. Much of the study of one-dimensional driven systems has been carried out in recent years in the field of ‘quantum chaos’ [4, 5]. In this field, one is looking for the manifestation of classical chaos in the quantum dynamics, and more generally for the relation between classical and quantum mechanics in the presence of chaos [6, 7]. One-dimensional driven systems provide some of the simplest examples of chaotic dynamics. However, in general, they are mixed systems, namely they are chaotic in some regions of phase space and regular

in others. While much progress has been made in the study of completely chaotic systems, the behaviour of mixed systems is still to a large extent an open problem.

The driven system which has been most widely studied in quantum chaos is the kicked rotor [8, 9]. This is a rigid planar rotor with an electric dipole, driven by a periodic train of pulses. For many years this was considered a pedagogical but non-physical example; however, recently an atomic system has been devised which interacts with laser fields, and which is described by the model Hamiltonian of the kicked rotor in some regimes of parameters [10, 11]. While classically the rotor absorbs energy from the field pulses indefinitely (in absence of friction), its quantum mechanical analogue absorbs energy only for a finite time and then saturates. This phenomenon, known as dynamical localization, is a result of interference effects in energy space, similar to Anderson localization on a one-dimensional random lattice, where a particle which diffuses classically localizes quantum-mechanically [12]. This analogy provides much of the present understanding of driven nonlinear systems. For the kicked rotor, the mapping to the Anderson model relies on the special spectral properties of the rotor ($E_n \propto n^2$), and on the sharp pulses of the driving field [8, 9].

In this work, we consider a class of driven systems which are different from the kicked rotor. This class is characterized by an unperturbed spectrum with a slowly varying density of states. Thus, a model with constant energy spacing in good approximation on some energy scale, and the deviations of the spectrum from harmonicity appear as an adiabatic change of this spacing as a function of energy. This slow dependence of the energy on quantum number also allows us to write the one period propagator as a product of a free propagator, that depends only on the action variable, and a ‘kick-like’ part that is weakly dependent on action. This ‘adiabatic nonlinearity’ of the spectrum is characteristic of many one-dimensional systems, of which several examples are the hydrogen-like atom, charge bubbles in liquid helium and surface electrons in a two-dimensional metal with a perpendicular magnetic field [13]. The last two examples have been described by the ‘bubble’ model [14–16]. The excitation of Rydberg atoms by microwave fields has been the subject of much study, both experimentally and theoretically [17–20]. This work presents a general framework for this class of systems, from which some new results emerge and some known results for special cases are confirmed.

We focus on the excitation of this class of systems by a monochromatic field, which is of high frequency compared to the typical frequencies of the system. This is always the situation for systems discussed in this work, for sufficiently high energy, since the density of states increases with energy. Thus, many unperturbed levels participate in the excitation (it is important, however, that the spectrum is discrete for our solution to hold). The basic idea employed in this work, is to solve the problem in a limited energy regime, where the locally defined energy spacing can be considered constant. The solution to the local problem relies on an exact solution of an integrable system. Then, we account for the large energy scales by exploiting the adiabatic dependence of parameters on energy. A similar idea was previously applied to the bubble model [21] using the special features of this system. In this work, the method is extended and the results are obtained under general conditions.

We study the excitation of the system by investigating its quasi-energy eigenstates, which are the stationary states of the periodic time-dependent problem [22, 23]. The exact eigenstates determine the quantum dynamics on all time scales, and finding them is as difficult as solving exactly the time-dependent Schrödinger equation. The advantage of this method is that sometimes one can characterize the eigenstates by general qualitative properties, from which qualitative properties of the dynamics may be deduced. This idea is inspired by the kicked-rotor analogy, where a robust qualitative feature of the quasi-energy

states—namely their exponential localization in energy space—explains the saturation in energy absorption. In the class of systems considered here, we find properties of the eigenstates on several energy scales. On the smallest scale, we find that, under quite general conditions, these states are composed of a ladder of sharp peaks, or ‘quasi-resonances’ (QR). These are related to quantum nearly resonant transitions between the energies of the undriven system and should be distinguished from the classical resonances. While the existence of such peaks is well known [24], here they are derived from the local non-perturbative solution, which enables the quantitative description of their location and shape. Surprisingly, the QR’s turn out to have a universal shape, independent of field parameters and of the details of the system. In particular, their width is independent of the driving field strength. These general results are presented in some detail in section 2. On a larger energy scale, our quantitative description of the local structure allows us to employ a method suggested by de Oliveira *et al* [16] for the study of the envelope of the eigenstates. The results of the local solution are used explicitly to construct a tight-binding equation for the amplitudes superimposed on the QR peaks. The parameters of this equation are non-universal and are related to the spectrum of the unperturbed system and to the matrix elements of the perturbation. In sections 3 and 4, the details of this equation for two examples are presented. The model consists of a particle in a potential well $V(x) = x^\sigma$, $x \geq 0$, driven periodically by a dipole electric field. In section 3, we consider the case of $0 < \sigma < 2$, of which the bubble model is a special case ($\sigma = 1$). We find that the envelope of the eigenstates exhibits a crossover from a localized to a more extended regime as a function of energy, and that their asymptotic tails have a power-law decay with a power which scales with the external field parameters. The scaling is independent of σ , and a crossover of the tail from normalizable to non-normalizable is found as a function of the external-field strength. This is a generalization of the result known for the bubble model [15, 16, 21]. In section 4 the case of $-2 < \sigma < 0$ is considered, of which the one-dimensional hydrogen atom ($\sigma = -1$) is a special case. Here, the potential well is singular at the origin, and this singularity is reflected in the equation for the amplitudes. In a certain energy range, the amplitudes are found to satisfy an Anderson-like model on a finite lattice. The scaling of the localization length with the field parameters depends on the power σ , and for the hydrogen atom the scaling found from the Kepler map is confirmed [25]. Section 5 is a summary and discussion of the results.

The method suggested here is general, and does not depend on peculiar features of a specific system. The local structure is universal for systems with an ‘adiabatically nonlinear’ spectrum; more precise conditions are given in section 2. The global structure depends on basic characteristics of the system, such as the analytic properties of the binding potential and the asymptotic behaviour of the matrix elements. The application of these results to similar systems which are not specifically discussed here, is therefore straightforward.

2. Local structure: quasi-resonances

In this section we show that for a wide class of smooth binding potentials, a bound particle driven by a high-frequency monochromatic force, exhibits a peak structure in energy in the quasi-energy eigenstates. These peaks, known as ‘photon states’ or QR, are related to one-photon excitation, and are thus separated by a distance of $\hbar\Omega$ in energy, where Ω is the driving frequency. The existence of approximate selection rules in the long-time transition amplitudes has been shown by Leopold and Richards [24]. In numerical calculations, sharp peaks have been observed for several systems in the quasi-energy states [14], in the long-time evolution [25] and in the spectrum of emitted radiation [26]. Here we calculate the precise form of the QR’s and formulate the conditions on the potential and on the driving

field for which these results are valid.

Consider the following one-dimensional Hamiltonian in action-angle variables (I, θ) :

$$\mathcal{H} = \mathcal{H}_0(I) + kV(I)g(\theta) \cos(\Omega t). \quad (2.1)$$

It is assumed that $\mathcal{H}_0(I)$ and $V(I)$ are smooth functions of I .

The Hamiltonian (2.1) is strictly time-periodic with a period of $T = 2\pi/\Omega$. Therefore, there is a set of good quantum numbers, the quasi-energies, associated with the stationary states of the system known as the quasi-energy states [22, 23]. These are the eigenphases and eigenstates of the Floquet operator, which is the evolution operator for one period. Thus we seek to solve the following eigenvalue equation:

$$\hat{U}|\psi_\lambda\rangle = e^{-i\lambda T}|\psi_\lambda\rangle \quad (2.2)$$

where \hat{U} is the Floquet operator and λ is the quasi-energy. Note that the quasi-energies are only defined by their residue modulo Ω . By convention, for a free system with levels E_n the quasi-energies are $E_n/\hbar \bmod \Omega$, so that in effect the spectrum is folded modulo $\hbar\Omega$.

It will be convenient to work in the basis of the eigenstates of \mathcal{H}_0 , which form a discrete non-degenerate spectrum labelled by the quantum number n :

$$\mathcal{H}_0|n\rangle = E_n|n\rangle. \quad (2.3)$$

The matrix elements of the Floquet operator in this basis will be calculated semiclassically. In the high-frequency regime ($\omega(I) \ll \Omega$, where $\omega(I) = \partial\mathcal{H}_0/\partial I$), it is natural to introduce the small parameter $\epsilon = \omega(I)/\Omega$. For the problems discussed in the present work, $\omega \rightarrow 0$ in the limit $I \rightarrow \infty$, so that ϵ is small in the region of high energy. Since the time scale relevant for quantum mechanics (the field period) is relatively short in this regime, the semiclassical approximation is expected to be very good and will be used throughout this paper. This does not imply, however, that the resulting properties of the system for high frequencies resemble those of the classical system better than the ones found for relatively low frequencies.

In this approximation

$$U_{n,n'} = \sum_\nu \mathcal{A}_\nu e^{\frac{i}{\hbar} S_\nu(n',n;T)} \quad (2.4)$$

where the sum extends over the classical trajectories ν connecting the action $n\hbar$ to the action $n'\hbar$ via a classical trajectory of time T . The trajectories differ from one another by the initial value of the unspecified variable, θ . The action $S_\nu(n',n)$ is the generating function of the canonical transformation of time evolution along the ν th classical trajectory; the amplitudes \mathcal{A}_ν are related to second derivatives of this action and incorporate some phase factors. An alternative way to express the semiclassical propagator is through an integral over initial values of θ . In this representation, the integrand is constructed such that the stationary phase approximation to the integral equals the sum (2.4). A precise derivation of this so-called ‘initial value representation’ is given by Levit and Smilansky [27]. They show that this representation is formally equivalent to the semiclassical approximation, which is the leading order in \hbar of the Feynman path integral. We employ a further approximation, namely we do not use the exact classical trajectories but only approximate ones. We assume that the action $n\hbar$ does not change much during the integration time T ; this is justified if the perturbing field is very small ($k \ll 1$), or alternatively if the integration time is very short compared to the natural period of the undriven system. Under these conditions, one may approximate the solution of $\theta(t)$ by $\theta + \omega(I)t$ where I is taken as some constant value

between $n\hbar$ and $n'\hbar$. For some models to be discussed in the next section this is formally a leading-order approximation in the high-frequency regime. The result is [28]

$$U_{n,n'} = e^{-i\frac{T}{\hbar}\mathcal{H}_0(n\hbar)} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-n')\theta - \frac{i}{\hbar}A(I,\theta)} d\theta \tag{2.5}$$

where the function $A(I, \theta)$ is defined by

$$A(I, \theta) = k \int_0^T V(I(t))g(\theta(t)) \cos(\Omega t) dt \tag{2.6}$$

which reduces to

$$A(I, \theta) = k \int_0^T V(I)g(\theta + \omega(I)t) \cos(\Omega t) dt \tag{2.7}$$

if the approximation $I = \text{constant}$, $\theta(t) = \theta + \omega t$ is used. This is always a leading order in the field strength k , and for the specific cases discussed in sections 3 and 4 it is also a leading order in the high-frequency limit (see appendix C). It is easily verified that the stationary points of this integral correspond to classical trajectories relating the action $n\hbar$ to $n'\hbar$ at time T . This approximation is sometimes called the ‘strong coupling correspondence principle’ (SCCP) [29].

An approximation for $A(I, \theta)$ in the high-frequency regime was developed in appendix A leading to:

$$A(I; \theta) = \begin{cases} 0 & 0 < \theta < 2\pi(1 - \epsilon) \\ \frac{2\pi}{\omega} kV(I)g_{m_+} \cos\left[\frac{2\pi - \theta}{\epsilon}\right] & 2\pi(1 - \epsilon) < \theta < 2\pi \end{cases} \tag{2.8}$$

where g_m are the Fourier coefficients of $g(\theta)$, and m_+ is the positive integer closest to $1/\epsilon$. It corresponds to the classical resonance, where $1/\epsilon = \Omega/\omega(I) = m_+$, which is an integer. (This is unrelated to the quasin resonances, which are of pure quantum origin.) It is shown in appendix A that the Fourier series of $A(\theta)$ is dominated by a few components near the resonant one with index m_+ , and the approximation (2.8) reproduces correctly these components. It is in general discontinuous and does not satisfy the continuity properties of the exact function, therefore it does not reproduce correctly the very high components. However, it is derived for a general function with coefficients g_m , that are smooth in m . Knowledge of the asymptotic properties of these coefficients in the large m limit can be used to restore the correct analytic properties of $A(I; \theta)$ and better approximations can be developed for specific models (see appendix A). Note that the more naïve approximation of keeping only the resonant term, or even a finite number of terms, always gives rise to an analytic function.

Semiclassically, the expression for the matrix element (2.5) can be interpreted as the product of two operators: a phase operator (diagonal in the $|n\rangle$ representation), and the operator $e^{-\frac{i}{\hbar}A(I;\theta)}$. Adiabaticity as a function of I implies that the latter effectively depends only on θ and the matrix element is just the $(n - n')$ Fourier component. The assumption is that $V(I)$ depends weakly on I and changes little in the region $I \in [n\hbar, n'\hbar]$. This assumption is commonly made (see, e.g. [30]). The product form can be derived also directly from the eigenvalue equation [21]:

$$\sum_{n'} U_{n,n'} \psi_{n'} = e^{-i\lambda T} \psi_n \tag{2.9}$$

where $\psi_n = \langle n | \psi_\lambda \rangle$ (explicit dependence of some quantities on λ is suppressed in what follows). It is convenient to introduce the translation operator in action space, $e^{i\hat{\theta}}$, defined by $e^{i\hat{\theta}} \psi_n = \psi_{n+1}$. With the help of this operator the equation can be written as

$$\sum_{r=-\infty}^{\infty} U_{n,n+r} e^{i\hat{\theta}r} \psi_n = e^{-i\lambda T} \psi_n \quad (2.10)$$

where $r = n' - n$. This is a good approximation in the regime of large n , since then the extension of the sum from $(-n)$ to $(-\infty)$ does not introduce a large error. Substituting the expression (2.5) for the matrix elements, the equation is

$$e^{-i\frac{T}{\hbar} \mathcal{H}_0(n\hbar)} \frac{1}{2\pi} \sum_r \int_0^{2\pi} d\theta e^{-ir\theta - \frac{i}{\hbar} A(I,\theta)} e^{i\hat{\theta}r} \psi_n = e^{-i\lambda T} \psi_n. \quad (2.11)$$

The idea is now to effectively separate the dependence of the elements in this equation on n from their dependence on r . The matrix elements are assumed here to vary slowly as a function of line number n , and fast as a function of $r = n' - n$. Thus we take n to be constant, and the translation operator is $\hat{\theta}$ conjugate to r . The operators (n and θ) may be commuted under this approximation, and the left-hand side of the equation may be summed to yield

$$e^{-i\frac{T}{\hbar} \mathcal{H}_0(n\hbar)} e^{-\frac{i}{\hbar} A(I;\hat{\theta})} |\psi_\lambda\rangle = e^{-i\lambda T} |\psi_\lambda\rangle \quad (2.12)$$

which is the same product of two operators. A simple intuitive explanation for this form is found from (2.8): the derivative $(\partial A / \partial \theta)$ represents the change in action during one period. Thus in our approximation most of the action transfer takes place in a limited region of θ . We will later consider two specific models of potentials in the half space $x \geq 0$; for these, over most of the particle's trajectory the potential is so smooth that the external field has very little effect other than modulating the dynamical variables. Near the point $x = 0$, the singularity of the potential induces transitions between different n states, and therefore the system is effectively 'kicked' at these points. This strong dependence on the angle θ enters through the function $A(I, \theta)$. This has been the basis for the construction of the classical Kepler map for the driven hydrogen atom [31, 25], and the similar impact map for the bubble model [15]. Note, however, that (2.12) is the approximate propagator for an external-field period and not an orbital period of the particle. Since the Kepler-type maps are constructed for an orbital period, which depends on energy and is not the natural period of the problem from a quantum-mechanical point of view, their quantization is not obvious [17].

Since $\mathcal{H}_0(n\hbar)$ varies slowly as a function of n , it can be expanded around a large value n_0 to first order in $l \equiv n - n_0$:

$$\mathcal{H}_0(n\hbar) \simeq \mathcal{H}_0(n_0\hbar) + \omega l \hbar \quad (2.13)$$

where $\hbar\omega = (\partial \mathcal{H}_0 / \partial n)|_{n=n_0}$. This effectively results in the expansion of the first operator in (2.12) to first order in $(n - n_0)$ (this is the first meaningful order in the expansion of the unperturbed Hamiltonian), and the second operator to zeroth order. The resulting eigenvalue equation appears as the equation for a 'linear kicked rotor' [32, 33], i.e. a rotor with an unperturbed harmonic spectrum, kicked at regular time intervals by a θ -dependent potential:

$$e^{i\omega l T} e^{-i\frac{i}{\hbar} A(n_0\hbar,\theta)} |\psi\rangle = e^{-i(\lambda - \lambda_0)T} |\psi\rangle \quad (2.14)$$

where $\lambda_0 = \mathcal{H}_0(n_0\hbar) / \hbar$. The quasi-energies of the linearized model are $\lambda^{lin} \equiv (\lambda - \lambda_0)$ and the kicking potential is given by the function A . The linear kicked rotor is an integrable,

exactly solvable system: closed expressions can be found for the eigenvalue spectrum and the corresponding eigenstates [32,33]. For this solution, it is convenient to express the kicking function in Fourier representation: $A(I, \theta) = \sum_m A_m(I) e^{im\theta}$. One finds that, for $1/\epsilon$ non-integer,

$$A_m(I) = \frac{kV(I)}{i\omega} \frac{mg_m}{m^2 - 1/\epsilon^2} (e^{im2\pi\epsilon} - 1). \quad (2.15)$$

The quasi-energy eigenstates can be written as $|\psi_j\rangle = e^{-i\lambda t}|u\rangle$, where $|u\rangle$ is a periodic function of time with period T . The exact functions for the linear kicked rotor are given by $\langle\theta|u\rangle = e^{i\phi(\theta)}$, with

$$\phi(\theta) = \mu\theta + \frac{1}{\hbar} \sum_{m \neq 0} \frac{A_m(n_0\hbar)}{(e^{-i2\pi m\epsilon} - 1)} e^{im\theta}. \quad (2.16)$$

The quantum number μ , characterizing the state, is related to the quasi-energy λ via

$$\lambda^{lin} T \equiv \mu\omega T \pmod{2\pi}. \quad (2.17)$$

In our case, substituting the Fourier components (2.15) to the general solution (2.16) yields

$$\phi(\theta) = \mu\theta - \frac{k}{i\hbar\omega} V(n_0\hbar) \sum_{m \neq 0} \frac{m}{m^2 - 1/\epsilon^2} e^{im(\theta+2\pi\epsilon)}. \quad (2.18)$$

This sum is calculated in appendix A. For symmetric coefficients g_m , (see (A.5))

$$\phi(\theta) = \mu\theta - \frac{\pi}{\hbar\omega} kV(n_0\hbar) \pi g_{m_+} \frac{\sin\left[\frac{\pi - \{\theta + 2\pi\epsilon\}}{\epsilon}\right]}{\sin(\pi/\epsilon)} \quad (2.19)$$

where $\{x\}$ denotes the residue of x modulo 2π . With (2.19) we may write the explicit form of the solution to (2.14) in the action representation, as a function of $l \equiv (n - n_0)$:

$$\langle l|u\rangle = e^{i\varphi} \sum_{m=-\infty}^{\infty} J_m(B) \text{sinc}[\pi(\mu - l + m/\epsilon)] \quad (2.20)$$

with the following definitions:

$$\begin{aligned} \varphi &= \pi(\mu - l)(1 - 2\epsilon) \\ B &= \frac{\pi}{\hbar\omega} kV(n_0\hbar) g_{m_+} / \sin(\pi/\epsilon) \end{aligned} \quad (2.21)$$

and $\text{sinc}(x) = \sin(x)/x$. In the high-frequency limit, $\epsilon \ll 1$, the function (2.20) is composed of a chain of peaks separated by approximately the energy of one photon, each weighted by an amplitude. Assuming that at each point tails of sinc functions centred far-away contribute incoherently, the absolute value of the wavefunction is large on this ladder of QR's. For an eigenstate corresponding to a quasi-energy λ^{lin} , the QR's are located at values l_j satisfying

$$\pi(\mu - l_j + m/\epsilon) = \pi\delta_j \quad (2.22)$$

with $|\delta_j| < \frac{1}{2}$, and μ related to λ^{lin} through (2.17). The positions of the peaks l_j at this wavefunction are expressed in terms of λ^{lin} as

$$E_{l_j} = l_j\hbar\omega = \lambda^{lin}\hbar + j\hbar\Omega - \hbar\omega\delta_j. \quad (2.23)$$

Thus, the quasi-energy λ^{lin} sets the origin of the QR ladder, and the peaks are located at approximately the equally spaced energies $j\hbar\Omega$, with a mismatch $\hbar\omega\delta_j$. The QR's are labelled by j , their position on this ladder. Figure 1 shows a graphic representation of the quantities l_j and δ_j which appear in (2.23): on an underlying lattice of spacing $\hbar\omega$ (the linearized unperturbed spectrum), is placed an incommensurate lattice of spacing $\hbar\Omega$ (the

photonic exact resonances). The origin of placing the photonic lattice is determined by the quasi-energy λ . Since the system has a discrete spectrum, it cannot be excited to an exact resonance but only to the closest available quantum state; this is l_j , which is the peak of the QR. The detuning at the j th peak is the mismatch between lattices, δ_j , which determines the exact shape of the QR.

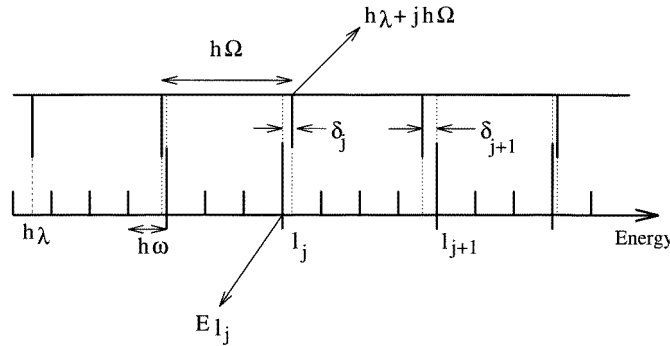


Figure 1. A graphic representation of (2.22) The ladder of exact resonances, with spacing $h\Omega$, is placed on top of the ladder of unperturbed levels with a local spacing $h\omega$. The location l_j and the detuning δ_j determine the properties of the j th peak, see (2.23).

Thus the sinc function around the j th QR can be written as

$$Q_j(l) = \frac{\sin[\pi(l - l_j + \delta_j)]}{\pi(l - l_j + \delta_j)}. \quad (2.24)$$

These functions form a non-orthogonal basis that is probably complete. For a given quasi-energy (2.23) implies that the corresponding eigenstate is composed of only a fraction of these. Note that the width of the QR is independent of the driving-field strength and of the density of unperturbed states. This is in contrast to the width associated with the transition rate given by Fermi's golden rule, and also in contrast to the Rabi width, both of which are invalid approximations in our regime of parameters. As will be shown in what follows, this property of the QR width is satisfied also by the eigenstates of the nonlinear problem.

The local linear solution provides an intuitive semiclassical understanding of the phenomenon of the QR's. As shown by Berry [34], the eigenstates of the linear kicked rotor can be obtained by a semiclassical quantization along classical invariant curves in phase space. The equation for such a curve is [34]

$$I_\mu(\theta) = \mu - \frac{1}{2} \sum_{m \neq 0} m A_m \frac{e^{im(\theta + \pi\epsilon)}}{\sin(m\pi\epsilon)}. \quad (2.25)$$

This curve is associated with a Floquet eigenstate corresponding to the quasi-energy λ^{lin} , which is related to μ via (2.17). Using an approximation similar to that of appendix A, the form of the curve is found to be

$$I_\mu(\theta) = \mu + kV(I) \frac{\pi g_{m_+}}{\omega\epsilon} \frac{\cos\left[\frac{\pi - (\theta + 2\pi\epsilon)}{\epsilon}\right]}{\sin(\pi/\epsilon)} \quad (2.26)$$

where $V(I)$ is almost constant in a large region. This curve exhibits details along the θ axis on a scale of ϵ . Since by quantization we wash out structures on areas smaller than \hbar in phase space, these details in θ can be reflected in the quantum eigenstates only at the cost of details in the n direction, namely by effectively sparsening the representation in n (see also

Jensen *et al* [35]). In the nonlinear system there are no invariant curves; however, because of the adiabatic change of parameters this still holds as an approximate local description.

On top of the QR structure, the state (2.20) has a typical width W in n , determined by the Bessel function:

$$W = \frac{\pi k}{\hbar \omega \epsilon} \frac{V(I) g_{m_{\pm}}}{\sin(\pi/\epsilon)}. \quad (2.27)$$

This is just the width of the corresponding classical curve (2.26) in the I direction. Beyond the region of allowed classical motion, the eigenstate decays in a way which depends on the analytic properties of $A(\theta)$ [33]. Thus, although the field strength k does not influence the shape of the QR, the width of the envelope as determined by the Bessel functions is proportional to k . It will be shown in the next sections that the field strength also determines the width of the envelope for the nonlinear system. If this envelope is power-law decaying, the power depends on k ; if it is exponentially localized, the localization length depends on k .

Recall that the exact solutions we have found so far are relevant to the linearized model, and are correct only off-resonance. We turn now to construct the eigenstates of the original nonlinear system. There are two cases to be considered, depending on the strength of the perturbation. If the perturbation is small, then off-resonance there exist invariant curves in phase space which can be approximated by (2.26). In this case, we conjecture that many eigenstates are bounded by such curves and can be approximated by eigenstates of the linearized equation. Since ϵ changes very slowly in phase space, neighbouring curves have almost the same value of ϵ and the corresponding eigenstates are almost orthogonal. Eigenstates corresponding to far away curves, for which ϵ is considerably different and orthogonalization may be destroyed, do not overlap much spatially, and moreover they are separated by regions where ϵ is resonant and our approximation breaks down. On the other hand if the perturbation is large and the linear eigenstates are so wide that they cover several resonances, then, in general, orthogonality between the linearized eigenstates can be destroyed while the spatial overlap is considerably large. Then it is expected that different regions can interact. Thus the two important length scales in phase space are the distance between classical resonances and the width of the linearized wavefunction. The ratio between these two scales determines the validity of a matching between different linearized eigenstates. It can easily be checked that, for the two models discussed in the next sections where the potential well is described by a power law, the condition for the validity of the matching is the same as the Chirikov criterion for resonance overlap [36]. This implies that for these models, matching between different solutions of the linearized problem is valid inside classically connected regions of phase space. Outside such regions, decay of the eigenstates is expected on classical grounds.

We turn now to describe the matching procedure between the regions where the linearization holds. The conditions for the validity of this procedure will be discussed for various specific cases in the following sections. We approximate the state in the vicinity of a given QR by the linear eigenstate with the local values of ω and λ_0 and with the given value of λ , and match different linearization regions. Using the approximation

$$E_{n_j} \approx E_{n_0} + (\partial E_n / \partial n)_{n_0} I_j \quad (2.28)$$

one finds that the quasi-resonances appear at values of $n = n_j$ satisfying

$$E_{n_j} = \lambda \hbar + j \hbar \Omega - (\partial E_n / \partial n)_{n_0} \delta_j. \quad (2.29)$$

The interpretation of this formula is similar to that of (2.23): the eigenvalue λ sets the origin of the ladder for the corresponding eigenstate, and the QR's are numbered by their

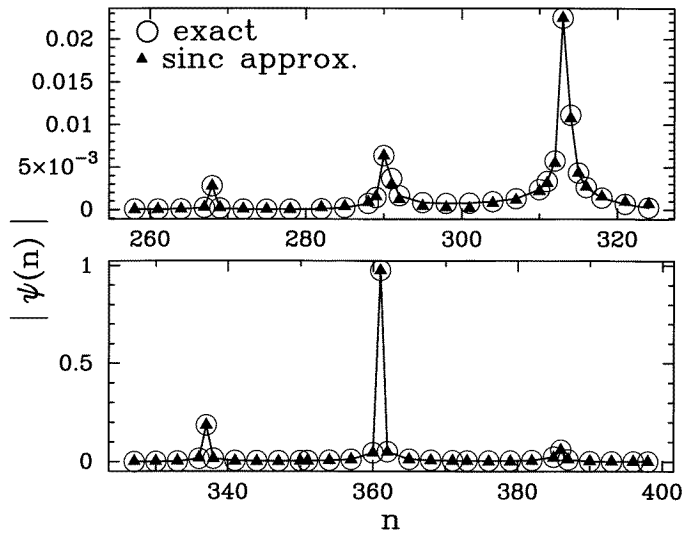


Figure 2. Numerical verification of (2.29). An exact Floquet eigenstate for the special case $\sigma = 1$ (open circles with full curve) is compared to a superposition of sinc shaped peaks (2.29) with parameters n_j and δ_j predicted by (2.28), and amplitudes fitted to the solution (full triangles).

position j on this ladder. Each is described approximately by the function $Q_j(n)$ of (2.24), characterized by the peak position n_j and the detuning $|\delta_j| < \frac{1}{2}$, both satisfying (2.29). This equation was obtained in a region where linearization holds around E_{n_0} . Now the centre of linearization can be varied, resulting in a local relation between E_{n_j} , j and δ_j . The matching of different linearized eigenstates may alter considerably the amplitudes A_j , therefore we write for the global eigenstate the following approximate form:

$$\psi_n = \sum_j A_j Q_j(n) = \sum_j A_j \frac{\sin[\pi(n - n_j + \delta_j)]}{\pi(n - n_j + \delta_j)} \quad (2.30)$$

where the amplitudes are yet to be determined. A numerical calculation verifying this approximation, with the analytically predicted values of n_j and δ_j for a specific system, is shown in figure 2. It is clear that the locations and shapes of the peaks are well described by this approximation.

Equation (2.30) together with (2.29) is the central result of this section. It is valid under the general conditions, (i) the unperturbed spectrum is ‘adiabatically nonlinear’, and (ii) the driving depends weakly on I and has smooth Fourier coefficients in θ . It implies that for the nonlinear system, the Floquet eigenstates are composed of a chain of sinc-shaped QR’s. As for the solution of the linear model, these QR’s are independent of the field parameters, in contrast to perturbative results. An heuristic explanation of this phenomenon can be given from semiclassical considerations. Since the transition between unperturbed states take place primarily around the origin $\theta = 0$, the typical time scale associated with these transition is the classical orbital period $2\pi/\omega$. Associating with this time scale an energy width via the uncertainty principle, and taking into account the unperturbed density of states, yields a constant width in n -space.

Equation (2.30) leaves open the determination of the amplitudes A_j . In the next sections, the information obtained here about the QR’s will be used to construct an equation for these amplitudes inside a region where matching is valid, following an idea of de Oliveira *et al*

[16]. This construction is system specific, and depends on the unperturbed spectrum and the matrix elements of the perturbation. It will be shown that, while the field parameters do not influence the shape of the single QR, they govern the behaviour of the overall envelope described by the amplitudes A_j .

3. A model with a positive power potential

In this section, we consider a particular model, of a particle of mass m in a one-dimensional potential well with the shape of a positive power law, driven by a monochromatic electric field via dipole interaction:

$$\mathcal{H}(x, p) = \frac{p^2}{2m} + bx^\sigma + kx \cos(\Omega t) \quad x \geq 0 \quad (3.1)$$

where $0 < \sigma < 2$, and $b > 0$. The transformation to action-angle variables gives for the undriven part

$$\mathcal{H}_0(I) = CI^\alpha \quad (3.2)$$

where C is a constant; we may choose the units such that $C = 1/\alpha$. The power α is given by

$$\alpha = \frac{2\sigma}{2 + \sigma} \quad (3.3)$$

satisfying $0 < \alpha < 1$. Therefore, in classical terms, the frequencies $\omega(I) = \partial\mathcal{H}_0/\partial I$ satisfy $\omega(I)_{I \rightarrow \infty} \rightarrow 0$. The rate of change of the classical frequency as a function of I also tends to zero in the same limit. When quantized, the energy levels of the unperturbed system are a slowly varying function of the quantum number $n = I/\hbar$. At high energies, they may be approximated by $E_n \approx \hbar\omega_{n_0}(n - n_0)$ with $\omega_{n_0} = (n_0\hbar)^{\alpha-1}$, which is a slowly varying function of n .

In order to express the full Hamiltonian in action-angle variables, one must write the position variable as a function of I and θ . As in the previous section, we make use of the fact that the wavefunctions are dominated by the Fourier components in the vicinity of the index closest to $1/\epsilon$. These are components of high frequency and consequently the wavefunctions are dominated by the region of the singularity in θ . It is shown in appendix B that semiclassically

$$x(I, \theta) = I^\gamma \sum_1^\infty g_m \cos(m\theta) \quad (3.4)$$

where

$$\gamma = \frac{2}{2 + \sigma} \quad g_m \sim_{m \rightarrow \infty} C_1/m^2. \quad (3.5)$$

It is important to notice that $\frac{1}{2} < \gamma < 1$, so that the driving also depends weakly on I . The full Hamiltonian in action-angle variables is, therefore,

$$\mathcal{H}(I, \theta) = \frac{1}{\alpha} I^\alpha + kI^\gamma g(\theta) \cos(\Omega t) \quad (3.6)$$

with α and γ defined by (3.3) and (3.5) respectively.

It was shown in the previous section, that the eigenstates of the Floquet operator are composed of QR's. We now discuss the regime of validity of (2.5), (2.7), called the SCCP, which lead to this result. It holds in the semiclassical limit whenever the classical trajectory can be approximated by $I = \text{constant}$, $\theta(t) = \theta + \omega(I)t$; thus, we now address the formal

question under what conditions this is a good description of the classical dynamics, for times of the order of the field period. One obvious condition is that the field is small enough; it is a leading order solution in k , the field strength. Sometimes, however, it is a leading order in a high-frequency expansion [21].

The dependence on the small high-frequency parameter ϵ is apparent in a particularly simple way, when writing the equations of motion in terms of the dynamical variables ϵ and θ and the dimensionless time $\tau = \Omega t$:

$$\frac{d\epsilon}{d\tau} = \epsilon^{\frac{2}{2-\sigma}} F_1(k, \Omega) \left(\frac{dg}{d\theta} \right) \cos(\tau) \quad (3.7)$$

$$\frac{d\theta}{d\tau} = \epsilon + \epsilon^{\frac{\sigma}{2-\sigma}} F_2(k, \Omega) g(\theta) \cos(\tau). \quad (3.8)$$

Since $\sigma > 0$, the time variation of ϵ is always slower than that of θ in the limit $\epsilon \rightarrow 0$, and therefore one can solve the equations in the Born–Oppenheimer approximation. The first step in this approximation consists in solving the equation for θ assuming ϵ is constant. If $\sigma > 1$, the leading order solution is $\theta(\tau) = \theta + \epsilon\tau$, and therefore the SCCP is formally a leading order in ϵ . For $\sigma \leq 1$, although the oscillatory term is not of higher order in ϵ , it is argued in appendix C that only the linear term contributes to the calculation of $A(I, \theta)$, (2.6). Therefore, the SCCP is not only a leading order in the field strength k , but also a leading order in the small parameter ϵ .

The asymptotic behaviour of the coefficients g_m ($\sim C_1/m^2$) can be used to calculate the function $A(I, \theta)$ for this specific model more accurately than the general formula (2.8):

$$A(I; \theta) = \begin{cases} 0 & 0 < \theta < 2\pi(1 - \epsilon) \\ \frac{C_1 2\pi k}{\Omega^2} I^{\frac{\sigma}{\sigma+2}} \left(\cos \left[\frac{2\pi - \theta}{\epsilon} \right] - 1 \right) & 2\pi(1 - \epsilon) < \theta < 2\pi \end{cases} \quad (3.9)$$

(see appendix A). Together with (2.5), this is to be compared with (4.7) of [21]. One can again solve the linearized model resulting in a solution of the form (2.20) with

$$B = \frac{k C_1 \pi I^{\frac{\sigma}{2+\sigma}}}{\hbar \Omega^2} \frac{1}{\sin(\pi/\epsilon)}. \quad (3.10)$$

The eigenstates of the Floquet operator in the unperturbed basis thus have the general form (2.30) with n_j and δ_j satisfying (2.29). The sequences of peak position n_j and detunings δ_j may be estimated using the approximation

$$\lambda \hbar + j \hbar = E_{n_j} + \left(\frac{\partial E_n}{\partial n} \right)_{n_j} \delta_j \cong E(n_j + \delta_j) = \frac{1}{\alpha} (\hbar(n_j + \delta_j))^\alpha. \quad (3.11)$$

Inverting this relation, one finds that

$$n_j = \text{int} \left\{ \frac{1}{\hbar} [\hbar \alpha (\lambda + j)]^{1/\alpha} \right\} \quad (3.12)$$

and

$$\delta_j = \text{frac} \left\{ \frac{1}{\hbar} [\hbar \alpha (\lambda + j)]^{1/\alpha} \right\}. \quad (3.13)$$

Since $1/\alpha > 1$, the sequence of neighbouring detunings δ_j is a pseudorandom sequence [37, 38]. Numerical verification of the form (2.30), with n_j and δ_j given by (3.12) and (3.13) respectively, was obtained for the special case of the bubble model ($\sigma = 1$) [21]. For this case, since there are closed expressions for the dipole matrix elements, it is possible to calculate the Floquet operator numerically to a high precision. In figure 2 a typical

eigenstate of this operator (open circles with full curve) is compared to a function of the form (2.30), with positions and detunings as calculated from (3.12) and (3.13), and arbitrary amplitudes. The good agreement persists throughout the whole basis used (400 n -states), though at very low values of n the agreement is less good. This is expected since in this region the semiclassical approximation does not hold.

The next step is to consider the structure of the eigenstate on a scale of many QR's. We first note that there are two important scales in n -space, the distance between QR's, $(1/\epsilon)$, and the bandwidth of the Floquet operator, which is the bound on classical motion in one period. This width may be estimated directly from Hamilton's equation for the action:

$$\frac{dI}{dt} = -kI^\gamma g'(\theta) \cos \Omega t. \quad (3.14)$$

Denoting by M the bound on the derivative $g'(\theta)$, one finds that

$$\left| \frac{dI}{dt} \right| \leq kMI^\gamma \quad (3.15)$$

so that the semiclassical bandwidth for transitions is

$$\Delta I \leq kMI^\gamma 2\pi / \Omega. \quad (3.16)$$

As argued by Richards *et al* [28], the quantum dynamics of the system is expected to be very different depending on the ratio between these two scales. If the distance between QR's is larger than the bandwidth, the absorption of even one photon involves a classically forbidden transition, and therefore if there are no special phase relations resulting in constructive interference the excitation is expected to be very limited. If there are many QR's inside one bandwidth, the excitation is expected to be more efficient. In our model, both scales vary as a function of n , and a crossover is found from the former to the latter type of behaviour as a function of n . The crossover point n_c scales with the parameters as

$$n_c \propto \left(\frac{\Omega^2}{k} \right)^{(2+\sigma)/\sigma}. \quad (3.17)$$

Thus, in general the wavefunctions are expected to be localized for small values of n and to crossover to a more extended behaviour at $n > n_c$. This is a local effect, which is easily traced in numerical computations of the wavefunctions. For the special case $\sigma = 1$ (the bubble model), the semiclassical Floquet matrix was calculated and diagonalized. The general structure described above was indeed verified, with the scaling (3.17) [21].

We now turn to investigate the long-range ($n \rightarrow \infty$) characteristics of the eigenstates. As explained in section 2, the condition for the validity of matching different regions where the linear solution holds, is that the width of the linear solutions (B/ϵ) is larger than the distance between classical resonances. For the model presented in this section we find this condition to be:

$$k > \left(\frac{2 + \sigma}{2 - \sigma} \right) \frac{\sin(\pi/\epsilon)}{\pi C_1} I^{\frac{2(\sigma-1)}{2+\sigma}}. \quad (3.18)$$

For typical non-resonant values of I , where $\sin(\pi/\epsilon)$ is of the order of 1, this inequality differs from Chirikov's criterion for resonance overlap [36] only by a constant. Therefore, the matching procedure presented in section 2 is valid where the Chirikov criterion is satisfied, namely in a classically connected region of phase space. It is seen that for $\sigma < 1$ local solutions can always be matched for high enough n , while for $\sigma > 1$ the matched solution only holds in a limited region, beyond which a decay of classical origin is expected. In a region where matching holds, we construct the equation for the amplitudes

of the QR's. The eigenvalue equation (2.2) can be equivalently written in an extended Hilbert space of variables (n, j) of which $|n, j\rangle = |n\rangle e^{-ij\Omega t}$ are basis functions [39–41, 23]. The corresponding operator in this space is the Floquet Hamiltonian, and the eigenvalue equation for it is

$$(E_n - \hbar j\Omega)\phi_{n,j} + \frac{k}{2} \sum_{n'} \langle n|\hat{x}|n'\rangle [\phi_{n',j+1} + \phi_{n',j-1}] = \hbar\lambda\phi_{n,j} \quad (3.19)$$

where $\phi_{n,j} = \langle n, j|\phi\rangle$ and the explicit dependence on λ was suppressed. Now using the formula for the eigenstates (2.30), one may approximate [21]

$$\phi_{n,j} \approx A_j Q_j(n) \quad (3.20)$$

and write the following equation for the amplitudes of the eigenstate associated with λ :

$$-A_j \hbar \delta_j \omega \frac{\sin(\pi \delta_j)}{\pi \delta_j} + \frac{k}{2} \sum_{n'} \langle n_j|\hat{x}|n'\rangle [A_{j+1} Q_{j+1}(n') + A_{j-1} Q_{j-1}(n')] \approx 0. \quad (3.21)$$

The dependence on λ enters through the functions $Q_j(n)$ and δ_j , see (2.24), (3.12), (3.13). This construction follows an idea of de Oliveira *et al* [16], for details see [21]. It results in a one-dimensional tight-binding equation for the amplitudes. The ‘diagonal potential’ is a function of the detuning δ_j at the j th QR; these are given by (3.13) and constitute a pseudo-random sequence. The hopping between QR's is given by the dipole matrix elements. It is shown in appendix B that, semiclassically,

$$\langle n|\hat{x}|n'\rangle \approx \frac{(n\hbar)^\chi C_1}{\omega^2(n-n')^2} \approx \frac{(n\hbar)^\chi C_1 \hbar^2}{(E_n - E_{n'})^2} \quad (3.22)$$

where $\chi = 2(\sigma - 1)/(\sigma + 2)$. The main contribution to the two sums over n' in (3.21) is from the vicinity of the neighbouring QR's, $(j + 1)$ and $(j - 1)$. It is assumed, as usual, that ω is a slowly varying function of n , so it is well defined in the region of n and n' . Using the approximation

$$\begin{aligned} E_n - E_{n'} &\approx -\hbar\Omega - (n' - n_{j+1})\omega\hbar & n' &\approx n_{j+1} \\ E_n - E_{n'} &\approx \hbar\Omega - (n' - n_{j-1})\omega\hbar & n' &\approx n_{j-1} \end{aligned} \quad (3.23)$$

the sums in (3.21) can be written in the following form:

$$\sum_{n'} \langle n_j|\hat{x}|n'\rangle Q_{j+1}(n') \approx \frac{C_1(n\hbar)^\chi}{\Omega^2} \sum_{n'} \frac{Q_{l+1}(n')}{[1 - (n_{j+1} - n')\epsilon]^2}. \quad (3.24)$$

It is shown in appendix F that for any smooth function \mathcal{F} (which can be expanded in a Laurent series),

$$\sum_M \mathcal{F}(M) \text{sinc}[\pi(M + \delta)] = \mathcal{F}(-\delta). \quad (3.25)$$

Therefore, the sums in (3.21) can be calculated to yield the following equation for the amplitudes:

$$\frac{-2}{C_1 \pi \sqrt{\alpha}} \left(\frac{\sin(\pi \delta_j)}{\sqrt{j}} \right) \left(\frac{\sqrt{\hbar} \Omega^{3/2}}{k} \right) A_j + \left[\frac{A_{j+1}}{(1 - \delta_{j+1} \epsilon_{j+1})^2} + \frac{A_{j-1}}{(1 + \delta_{j-1} \epsilon_{j-1})^2} \right] \approx 0. \quad (3.26)$$

The one-dimensional tight-binding equation for the amplitudes A_j now contains only nearest-neighbour coupling. The diagonal potential is given by the pseudo-random function $\sin(\pi \delta_j)$, damped by a factor of \sqrt{j} . The hopping terms are weakly dependent on position, and tend to unity in the limit $\epsilon_j \rightarrow 0$. For the case of a truly random potential and strictly constant hopping, the eigenstates are power localized with a power proportional to the square

of the prefactor of the diagonal potential [42]. Numerical calculations for the model (3.26) with $\sigma = 1$, provide evidence that these differences do not alter the qualitative conclusion of the theorem (Brenner and Fishman [21] following de Oliveira *et al* [16]). Thus, we may conclude that the envelope of the wavefunction decays asymptotically with a power law as a function of the QR number j , and that this power scales as $(\hbar\Omega^3/k^2)$. Therefore, there is a critical value k_c of the external field, for which the power of the wavefunctions are larger than $(-\frac{1}{2})$ and consequently they are non-normalizable. This critical field depends on parameters as $k_c \propto \sqrt{\hbar\Omega^3}$, independent of σ , with a prefactor which cannot be determined from this calculation. This crossover was first found by Benvenuto *et al* [15] for the special case of the bubble model, $\sigma = 1$, by using the classical impact mapping.

In conclusion, we have found in this section that for the positive power-law model (3.1), one may characterize the wavefunctions of the Floquet operator by two scales which include many QR's: on a local scale a crossover is found from localized to more extended behaviour as a function of n . The point of crossover between the two regime scales as $n_c \propto (\Omega^2/k)^{(2+\sigma)/\sigma}$. In the longer range, i.e. in the asymptotic limit $n \rightarrow \infty$, the behaviour is found to be a power-law decay, with the power proportional to $(\hbar\Omega^3/k^2)$, independent of σ .

4. A model with a negative power potential

In this section, we consider a class of models similar to those discussed in the previous section, but with a binding potential described by a negative power (thus having a singularity at the origin). The driving is a harmonic electric field treated in the dipole approximation. It will be convenient to work in a different gauge for this problem,

$$\mathcal{H}(x, p) = \frac{1}{2m} \left[p + \frac{k}{\Omega} \cos(\Omega t) \right]^2 - bx^\sigma \quad x \geq 0 \quad (4.1)$$

where $b > 0$. We consider the region $-2 < \sigma < 0$, so that bound states exist in the potential well [30]. The special case of $\sigma = -1$ corresponds to the one-dimensional potential of hydrogen like atoms. This is an approximation the three-dimensional motion of a particle in a coulomb binding potential with high energy and small angular momentum [3].

Similar to the previous section, we use action-angle variables of the unperturbed system; in these variables and in appropriate units,

$$\mathcal{H} = \frac{1}{\alpha} I^\alpha + \frac{k}{\Omega} p(I, \theta) \cos(\Omega t). \quad (4.2)$$

The power α is related to σ by (3.3) and is in the interval $(-\infty, 0)$, so the bound part of the spectrum is bounded from above by $E = 0$. We consider the dynamics of this system only inside the bound part of the spectrum, i.e. we ignore transition into the continuum. This is a good approximation for the dynamics if the initial condition is far enough from the continuum, for time scales over which the system does not evolve to states too near the continuum [25, 43].

The use of the SCCP for calculating the Floquet matrix elements, (2.5), is justified here either as a leading order in the field strength k , or as a leading order in ϵ , as explained in appendix C. The function $A(\theta)$ is calculated in appendix A:

$$A(\theta) = \begin{cases} 0 & 0 < \theta < 2\pi(1 - \epsilon) \\ \frac{C_2 2\pi k}{\Omega^\eta} \sin \left[\frac{2\pi - \theta}{\epsilon} \right] & 2\pi(1 - \epsilon) < \theta < 2\pi \end{cases} \quad (4.3)$$

where $\eta = (4 - \sigma)/(2 - \sigma)$. The Fourier components of p decay as $g_m \sim C_2/m^{2/2-\sigma}$ (see appendix D). This function has a similar structure to the one calculated in section 2, and the reason for its form is similar: the effective energy transfer takes place in the vicinity of the origin. In appendix E, we present an alternative way to calculate this function for the case of the hydrogen atom. In this case the exact transformation to action angle variables is used, and the result (4.3) is derived as the leading order in the asymptotic expansion in $\epsilon \rightarrow 0$.

Using this function it is possible to proceed as explained in section 2, to find that the eigenstates of the local linearized problem have the form of (2.20). In this case the properties of the g_m 's are such that

$$\begin{aligned}\varphi &= \pi(\mu - l)(1 - 2\epsilon) - \frac{C_2 k \epsilon}{\hbar \Omega^\eta} \\ B &= \frac{C_2 \pi k}{\hbar \Omega^\eta \sin(\pi/\epsilon)}.\end{aligned}\quad (4.4)$$

The location of the QR's for the linearized model are given by (2.23). Since the energies are negative and bounded by zero, there is a finite number of QR's. Denoting by r_{max} the maximal value of r , the location of the QR's is

$$E_{n_r} = \hbar\lambda - \hbar\Omega r_{max} + \hbar\Omega r - \delta_r \hbar\omega. \quad (4.5)$$

The shift $\hbar\Omega r_{max}$ was introduced in order to express the negative energies E_{n_r} in terms of the positive index r . The sequence of detunings δ_r is calculated as in the previous section, to yield

$$\delta_r = \text{frac} \left\{ \frac{1}{\hbar} [\alpha(\hbar\lambda - (r_{max} - r)\hbar\Omega)]^{1/\alpha} \right\}. \quad (4.6)$$

Recall that $-\infty < 1/\alpha < 0$. By definition, $(r_{max} - r)$ is always positive and it decreases as r increases, therefore it is clear that the sequence δ_r becomes more irregular for large values of r tending to r_{max} . Figure 3 shows a typical sequence of detunings δ_r for the hydrogen atom, where $\alpha = -2$, and it is seen that indeed at larger values of r the sequence is more irregular. The straight lines connecting consecutive points at small values of r indicate that in this region the detunings behave quasiperiodically. We may estimate the

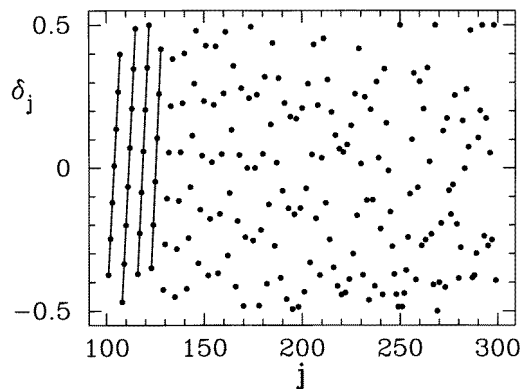


Figure 3. Typical sequence of neighbouring detunings δ_r for the hydrogen atom ($\sigma = -1$). The lines connecting consecutive elements at low values of r indicate that the sequence is regular. For larger values of r it is more irregular.

point at which the sequence becomes irregular to be the point where the difference between the values of the terms in the curly brackets in (4.6) for neighbouring sequence elements is of the order of 1. Using the approximation $E_{n_r} \approx \hbar\lambda - \hbar\Omega(r_{max} - r)$, one finds that the condition for this is $\epsilon < 1$.

We now turn to investigate the long-range characteristics of the eigenstates. The condition for matching in this case, corresponding to (3.18)

$$k > \left(\frac{2 + \sigma}{2 - \sigma} \right) \frac{\sin(\pi/\epsilon)}{C_2\pi} \Omega^{\frac{\sigma}{2-\sigma}} I^{\frac{3\sigma-2}{2+\sigma}}. \quad (4.7)$$

For typical non-resonant values of I , where $\sin(\pi/\epsilon)$ is of the order of 1, this inequality differs from Chirikov's criterion for resonance overlap only by a constant. Therefore, the matching procedure is valid where the Chirikov criterion is satisfied, namely in a classically connected region of phase space. It is seen that since σ is negative, phase space is always connected for large enough values of I .

The equation for the amplitudes of the QR's is constructed in the same way as in the previous section. We start from the analogue of (3.21), with the coupling given by the momentum operator:

$$-A_r \hbar \delta_r \omega \frac{\sin(\pi \delta_r)}{\pi \delta_r} + \frac{k}{\Omega} \sum_{n'} \langle n_r | \hat{p} | n' \rangle [A_{r+1} Q_{r+1}(n') + A_{r-1} Q_{r-1}(n')] \approx 0. \quad (4.8)$$

The matrix elements of the momentum operator are estimated semiclassically in appendix D, and one may write the result as

$$\langle n_r | \hat{p} | n' \rangle \approx i\omega C_2 \hbar^{\frac{2}{2-\sigma}} \frac{\text{sgn}(n_r - n')}{|E_{n_r} - E_{n'}|^{\frac{2}{2-\sigma}}}. \quad (4.9)$$

Substituting this expression into the sum over n' and using (3.23), one has for the first sum

$$\frac{k}{\Omega} \sum_{n'} \langle n_r | \hat{p} | n' \rangle Q_{r+1}(n') \approx \frac{k i \omega C_2}{\Omega^\eta} \sum_{n'} \frac{Q_{r+1}(n') \text{sgn}(n_r - n')}{|1 - (n_{r+1} - n')\epsilon|^{\frac{2}{2-\sigma}}}. \quad (4.10)$$

Expanding the denominator around $n' = n_{j+1}$ and summing according to appendix F, one finds the following equation:

$$\left(\frac{\hbar \Omega^\eta}{\pi C_2 k} \right) \sin(\pi \delta_r) A_r + \left[\frac{i A_{r+1}}{(1 - \delta_{r+1} \epsilon_{r+1})^{\frac{2}{2-\sigma}}} + \frac{-i A_{r-1}}{(1 + \delta_{r-1} \epsilon_{r-1})^{\frac{2}{2-\sigma}}} \right] \approx 0. \quad (4.11)$$

For $\epsilon < 1$, the first term, namely the diagonal potential, may be considered similar to a random one. Neglecting the weak dependence of the hopping terms on r , and defining $\tilde{A}_r = e^{i\pi r/2} A_r$, one obtains a one-dimensional Anderson model for the amplitudes \tilde{A}_r . The eigenstates of this model have an absolute value which is exponentially localized with a perturbative estimate for the localization length given by

$$\xi \sim \frac{k^2}{\hbar^2 \Omega^{2\eta}}. \quad (4.12)$$

Figure 4 shows a numerical calculation of the localization length as the inverse Lyapunov exponent corresponding to equation (4.11) for the case of the hydrogen atom in the region $\epsilon < 1$, using the transfer-matrix technique. It is seen that the estimate (4.12) agrees well with the numerical result. In this case $\eta = \frac{5}{3}$, and the scaling of the localization length with the field parameters is the same as that obtained from the Kepler map [25].

In conclusion, we have shown in this section that for the negative power-law model, in the high-frequency regime ($\epsilon < 1$) the equation for the amplitudes superimposed on

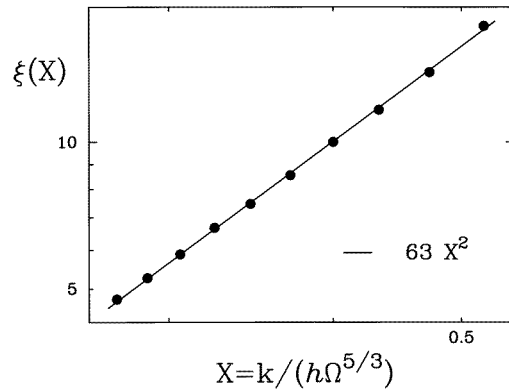


Figure 4. Localization length for equation (4.11) with $\sigma = -1$ (the hydrogen atom), in the region $\epsilon < 1$. The calculation was performed using the transfer-matrix technique, and identifying the inverse Lyapunov exponent with ξ .

the peak structure is an Anderson-like tight-binding model. We have completely neglected the effect of the continuum on the dynamics; this should be taken into account once the localization length is large enough so that there is considerable probability to find the system near $E = 0$ [25].

5. Summary and discussion

In this paper, we considered the excitation of a quantum system with an adiabatically nonlinear spectrum by a high-frequency field. This was done by investigating the properties of the Floquet eigenstates, which are the stationary states of the time-dependent system. These properties can be characterized according to the energy scale considered. The properties found on a small energy scale are universal for this class of systems, while the properties on a large energy scale are system specific.

On a small energy scale, the Floquet eigenstates were found to be composed of a ladder of sharp peaks, the quasi-resonances QR's. While their existence was already well known, the details of their shape and width were revealed in the present work. It was found that the QR's have a universal shape of a sinc function, (2.24), independent of the driving-field parameters and of the details of the system. This result was obtained by solving an exact model, the linear kicked rotor, which approximates the system in a finite-energy regime. The constant line-width can be simply related by uncertainty to the typical time for energy transfer, which in these systems is not the field period, but the orbital period of the undriven system. Thus the eigenstate can be described by (2.30) with the details of the sinc functions predicted by (2.29).

The behaviour of the eigenstates over a large energy scale was obtained by constructing an effective equation for the QR amplitudes. For this purpose, the universal properties of the QR's on small energy scales were used explicitly. For a monochromatic field, this turns out to be a nearest-neighbour tight-binding equation in one dimension, whose parameters can be directly related to the properties of the physical system. The diagonal potential is related to the unperturbed spectrum, while the hopping terms are related to matrix elements of the perturbation between neighbouring QR's. If the matrix element of the interaction

with the driving field \hat{O} in the unperturbed basis can be written semiclassically in the form

$$\langle n|\hat{O}|n'\rangle = V(n\hbar)g_{n-n'} \quad (5.1)$$

as in the case for the systems studied in the present work, then the most general form of the tight-binding equation is

$$-\frac{\hbar\omega}{\pi} \sin(\pi\delta_j)A_j + \frac{k}{2}V(n_j\hbar) \left[\tilde{g} \left(-\frac{1}{\epsilon_{j+1}} + \delta_{j+1} \right) A_{j+1} + \tilde{g} \left(\frac{1}{\epsilon_{j-1}} + \delta_{j-1} \right) A_{j-1} \right] \approx 0 \quad (5.2)$$

where $\tilde{g}(x)$ is the extension of the discrete function g_m to a continuous variable. The index j labels the QR number, which occurs at the unperturbed state n_j . The amplitude at this QR is denoted by A_j , the detuning from exact resonance by δ_j and the local unperturbed frequency is ω .

We considered in detail the model of a particle in a potential well described by a power law, $V(x) \sim x^\sigma$, driven by an electric field in the dipole approximation. For positive powers, $0 < \sigma < 2$, the eigenstate has two qualitatively different regimes in n -space. In the small n regime it is exponentially localized, while for larger values of n it is more extended, with a power-law asymptotic decay. The crossover between the two regions takes place at n_c , satisfying

$$n_c \sim (\Omega^2/k)^{(2+\sigma)/\sigma}. \quad (5.3)$$

This is a generalization of a result previously found for the special case of the bubble model, $\sigma = 1$ [21]. In the asymptotic regime $n \rightarrow \infty$, the special case of (5.2) is similar to the Anderson model in an electric field. The diagonal potential is a pseudorandom sequence whose details depend on σ , but otherwise this equation—and thus the asymptotics of the eigenstates—is independent of σ . The eigenstates are found to be power-law decaying,

$$A_j \sim 1/j^{(\hbar\Omega^3/k^2)}. \quad (5.4)$$

This implies the existence of a crossover field strength $k_c \sim \sqrt{\hbar\Omega^3}$ beyond which the tails of the eigenstates turn non-normalizable. A result of this general form was announced by de Oliveira *et al* [16]. For the special case of the bubble model, this result was first obtained by use of a Kepler-like map [15]. The reason that in this regime the result is independent of σ is related to the fact that the asymptotic behaviour of the dipole matrix elements is similar for all σ , which is a result of the triangle-like singularity that all these potentials have at the origin.

For negative powers, $-2 < \sigma < 0$, the effective amplitude equation is similar to the Anderson model on a finite lattice, due to the finite-energy regime of the bound spectrum. However, only in the high-frequency regime, $\omega < \Omega$, is the diagonal potential pseudorandom. In this regime the localization length is

$$\xi \sim k^2/\hbar^2\Omega^{2\eta} \quad (5.5)$$

with $\eta = (4-\sigma)/(2-\sigma)$. For the special case of the hydrogen atom, $\sigma = -1$, this reduces to the scaling found from the Kepler map [25]. The singularity of the binding potential in this region is σ -dependent, thus the asymptotic behaviour of matrix elements depends on σ , leading to a dependence on the details of the amplitude equation and of the localization length.

Equation (5.2) can be applied also to other systems with an adiabatically nonlinear spectrum. For various systems and different driving forces, other tight-binding models may be found. Since the behaviour is dominated by the form of the potential near the origin the

results of this work are also applicable to other potentials as long as they behave like x^σ near the origin, while their behaviour in other regions may be quite different, provided it is smooth. Note that the results follow from the large- m behaviour of Fourier elements of the dynamical variables, which depend on the type of singularity of the binding potential at the origin. The Fourier components which are important for our solutions are those with index in the vicinity of $m \approx 1/\epsilon$. A smoothing of the binding potential in a small region will only change the behaviour of the Fourier components asymptotically; if this region is small enough, then the change will be considerable only at Fourier components much higher than $1/\epsilon$ and therefore will not change our results.

In this paper several approximations were made which require some further justification. These were supported by heuristic arguments outlined in the paper and comparison of the results with exact numerical calculations for specific systems. The most important of these is the matching procedure between various regions where the linearization holds. The relation of its validity to the existence of extended chaos in the corresponding classical system requires some further clarification. Another approximation is that in calculating the Floquet operator, only the linear average change of the angle variable as a function of time contributes. This is formally a leading order in the field strength, and for the models considered in sections 3 and 4 it was shown to be also a leading order in ϵ . For a general system driven by a high-frequency field it requires further justification. These problems are left for further studies.

Acknowledgments

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Appendix A. Approximate summation of resonant sums: calculation of $A(\theta)$

In this appendix we present an approximate way to calculate the function $A(\theta)$ which appears in the matrix elements of the Floquet operator (see (2.7), (2.12)). Inserting the Fourier expansion of g ,

$$g(\theta + \omega t) = \sum_m g_m e^{im(\theta + \omega t)} \quad (\text{A.1})$$

and performing the integral dt , one finds

$$A(I; \theta) = \frac{kV(I)}{i\omega} \sum_{m \neq 0} m g_m \frac{e^{i2\pi\epsilon m} - 1}{(m^2 - 1/\epsilon^2)} e^{im\theta}. \quad (\text{A.2})$$

If the coefficients, g_m , are symmetric (as is the case if interaction is a function of position x only),

$$A(I; \theta) = \frac{2kV(I)}{\omega} \left\{ \sum_{m=1}^{\infty} \frac{m g_m}{(m^2 - 1/\epsilon^2)} \sin[m(\theta + 2\pi\epsilon)] - \sum_{m=1}^{\infty} \frac{m g_m}{(m^2 - 1/\epsilon^2)} \sin[m(\theta)] \right\}. \quad (\text{A.3})$$

If g_m is a smooth function which does not decay too rapidly, then in the high frequency limit, $\epsilon \ll 1$, the sum is dominated by the terms m_{\pm} . These are the terms with indices closest in absolute value to $1/\epsilon$, so that the denominator in the sum of (2.18) is minimal. They satisfy $m_{\pm} \approx \pm 1/\epsilon$ which corresponds to local classical resonances $\Omega \approx \pm(m_{\pm}\omega)$. If g_m is too rapidly decreasing, the sum will have terms comparable to the resonant ones at small values of m . Denoting by Δ the detuning from the classical resonance, ($\Delta = 1/\epsilon - m_{+}$), the following condition on the decay of g_m should hold in order for the resonant terms to be dominant:

$$\frac{g_{m_{+}}}{g_1} > 2\epsilon^2 \Delta. \tag{A.4}$$

The resonant terms appear at indices m_{\pm} of large absolute value and give two regions of contribution to the sum which are well separated from each other. In ordinary ‘resonance approximations’, the sum would be approximated by these two leading terms [28]. However, we may use the smoothness of g_m in order to take into account correctly the contribution of many terms around the resonant ones:

$$\sum_{m=1}^{\infty} \frac{m g_m}{(m^2 - 1/\epsilon^2)} \sin[m\theta] \approx g_{m_{+}} \sum_{m=1}^{\infty} \frac{m}{(m^2 - 1/\epsilon^2)} \sin[m\theta] = g_{m_{+}} \frac{\pi \sin[\pi - \{\theta\}/\epsilon]}{2 \sin[\pi/\epsilon]} \tag{A.5}$$

where $\{\theta\}$ denotes the residue of θ modulo 2π . The last equality is a result of exact summation. Substituting the approximation (A.5) in (A.3), one finds the result (2.8). Figure A1 shows that the approximation of (A.5) is indeed a good one. Our approximation is least good at the edges of the interval $[0, 2\pi]$, since there the values of $\cos(m\theta)$ do not oscillate very much as a function of m , and a large contribution may be given by the tails. At other values of θ the cosines oscillate so that the main contribution to the sum indeed comes from the close vicinity of the resonant term. The relative error between the absolute values of exact calculation and our approximation is about 0.15.

For the positive power-law model with dipole driving presented in section 2, the asymptotic behaviour of the coefficients is $g_m \sim C_1/m^2$ (see appendix B). Since the resonant sum is dominated by the vicinity of m_{+} (that is large), we may use this asymptotic form to

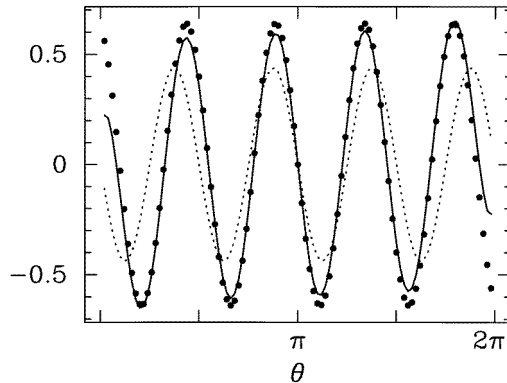


Figure A1. The resonant series in equation (A.5) as a function of θ . The exact sum of the LHS (full curve), compared with the approximation RHS (full circles). In this example, $g_m = m^{-2/3}$. For comparison, the dotted curve shows the contribution of the resonant term m_{+} .

calculate the sums more accurately. Using the formula

$$\sum_{m=1}^{\infty} \frac{\sin(m\theta)}{m(m^2 - 1/\epsilon^2)} = \frac{\epsilon^2}{2} (\{\theta\} - \pi) + \pi \frac{\epsilon^2}{2} \frac{\sin[\pi - \{\theta\}/\epsilon]}{\sin[\pi/\epsilon]} \quad (\text{A.6})$$

one finds the expression (3.9) for A .

For the negative power-law model presented in section 3, the coupling is linear in p . Thus the coefficients g_m are antisymmetric and the analogue of (A.3) is

$$A(I; \theta) = \frac{2kV(I)}{\omega\Omega i} \left\{ \sum_{m=1}^{\infty} \frac{mg_m}{(m^2 - 1/\epsilon^2)} \cos[m(\theta + 2\pi\epsilon)] - \sum_{m=1}^{\infty} \frac{mg_m}{(m^2 - 1/\epsilon^2)} \cos[m(\theta)] \right\}. \quad (\text{A.7})$$

It is shown in appendix D that the Fourier components of p decay as $g-m \sim C - 2/m^{2/2-\sigma}$, which is a power of m between (-1) and $(-\frac{1}{2})$. Therefore, the product mp_m varies slowly inside the sums of cosines, and one may write the approximation analogous to (A.5) for this case:

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{mg_m}{(m^2 - 1/\epsilon^2)} \cos[m\theta] &\approx m_+ g_{m_+} \sum_{m=1}^{\infty} \frac{\cos[m\theta]}{(m^2 - 1/\epsilon^2)} \\ &= m_+ g_{m_+} \left[\frac{\epsilon^2}{2} - \frac{\pi\epsilon}{2} \frac{\cos[\pi - \{\theta\}/\epsilon]}{\sin[\pi/\epsilon]} \right]. \end{aligned} \quad (\text{A.8})$$

Using the asymptotic properties of the coefficients g_m and $V(I)$ for the momentum, given in appendix D, one finds the approximation (4.3) for A .

Appendix B. Matrix elements for $\sigma > 0$

In this appendix, the dipole matrix elements will be calculated semiclassically for the model (3.1). In the semiclassical limit, the dipole-matrix element between states $|n\rangle$ and $|m\rangle$ is just the $(n - m)$ Fourier component of the variable $x(I, \theta)$. The relation to the angle variable is:

$$\theta(x, I) = \frac{\partial S}{\partial I} = \frac{\partial E}{\partial I} \int^x \frac{m}{\sqrt{2m(E - by^\sigma)}} dy \quad (\text{B.1})$$

where $E = \mathcal{H}_0(I)$. The function $x(I, \theta)$ can be obtained by inverting this relation. Since $\theta(x, I)$ is defined in terms of the integral of a smooth function, it is differentiable with a derivative

$$\frac{\partial \theta}{\partial x} = \frac{\partial E}{\partial I} \frac{m}{\sqrt{2m(E - bx^\sigma)}}. \quad (\text{B.2})$$

This derivative is non-vanishing at all points (since the potential is everywhere finite), therefore the implicit equation may be inverted, for each branch of the square root. Note that at the turning point x_0 defined by $E = bx_0^\sigma$, this derivative diverges therefore the derivative of the inverse function $x(I, \theta)$ is zero. Thus if we define the angle θ to be zero at the origin and π at this turning point, then $x(I, \theta)$ is a continuous function of θ , symmetric with respect to π , with a finite derivative everywhere which vanishes only at $\theta = \pi$. Since x is always positive (the potential is defined on a half space), necessarily the function $x(I, \theta)$ has a discontinuous derivative at endpoints of the interval $[0, 2\pi]$. The function $x(I, \theta)$ was calculated numerically for several values of σ . Some results are shown in figure A2. It is seen that the general form of the function is qualitative as described above. For $\sigma = 1$ it is an inverted parabola, while for $\alpha = 2$ it is proportional to $\sin(\theta/2)$.

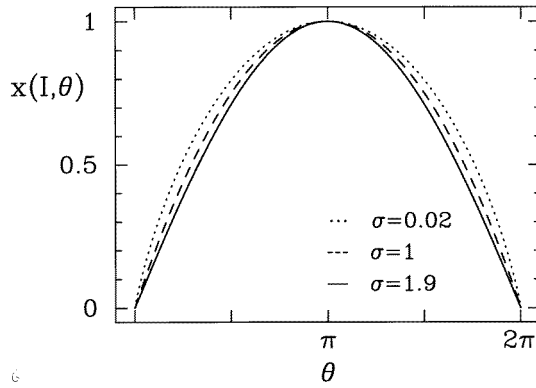


Figure A2. The classical variable $x(I, \theta)$ for various values of σ . For comparison, all graphs are scaled so that at the turning point $x = 1$.

These properties of the function $x(I, \theta)$ imply that its Fourier series has elements g_m which decay as $1/m^2$. If one is interested only in the high Fourier components, as we are in this paper, the dependence of x on the action I can be obtained from calculating the integral in the vicinity of $x = 0$ (where $\theta = 0$ and the function is singular). In this region the generating function can be approximated by a linear function,

$$S(x; I) = \int^x \sqrt{2m(E(I) - by^\sigma)} dy \approx \sqrt{2mE(I)}x. \tag{B.3}$$

Its derivative with respect to I is $\theta(x; I)$ to first order in x , which can be inverted to give $x(I, \theta)$ to first order in θ :

$$x(I, \theta) \propto I^{2/(2+\sigma)}\theta. \tag{B.4}$$

Now $x(I, \theta)$ can be approximated to be a function of I multiplied by a function of θ . In this approximation (which reproduces correctly the high Fourier components) the position variable is

$$x(I, \theta) = I^\gamma \sum_m g_m \cos m\theta \tag{B.5}$$

with $\gamma = 2/(2 + \sigma)$ and $g_m \sim C_1/m^2$ for large m . The semiclassical matrix elements satisfy:

$$\langle n|\hat{x}|m\rangle \propto \frac{(n\hbar)^\gamma}{(n - m)^2}. \tag{B.6}$$

For the special case of the bubble model, $\sigma = 1$, one has (B.6) with $\gamma = \frac{2}{3}$ which agrees with the exact result in the limit $n \gg 1$.

The next term in the expansion around $x = 0$ in (B.3), gives

$$x(I, \theta) \sim I^{2/(2+\sigma)}(a\theta - b\theta^{1+\sigma}) \tag{B.7}$$

where a and b are constants. This shows that also in the next order in the expansion the product form of $x(I, \theta)$ is preserved. It leads to a correction to the Fourier components in (B.5), $g_m \sim C_1/m^2 + C'_1/m^{2+\sigma}$, which renormalizes the function $A(I, \theta)$ by a factor of $(1 + (C'_1/C_1)\epsilon^\sigma)$. Thus, the conclusions of section 2 remain qualitatively unchanged to the next order in ϵ .

Appendix C. The SCCP as a high-frequency approximation

In this appendix, we show that for the two models discussed in sections 3 and 4, the calculation of the Floquet operator using the SCCP is a leading order in the limit $\epsilon \rightarrow 0$.

Consider the dimensionless equations of motion for the positive power-law model, (3.7), (3.8). As explained in section 3, the first step in the time-scale separation is to solve the equation for θ assuming that ϵ is constant. For $\sigma > 1$, the leading-order solution is $\theta = \theta_0 + \epsilon\tau$, therefore the SCCP is clearly a leading order in ϵ . For $\sigma < 1$ the oscillatory term in the equation is dominant in the limit $\epsilon \rightarrow 0$. Here we consider the latter case, and in the regime where $\epsilon \ll 1$, we solve the equation for θ by self-consistent iterations.

For $\epsilon = 0$ the solution is simply $\theta = \theta_0$. Inserting this back into the equation of motion,

$$\frac{d\theta}{d\tau} = \epsilon + \epsilon^{\frac{\sigma}{2-\sigma}} F_2(k, \Omega) g(\theta_0) \cos(\tau) \quad (\text{C.1})$$

which gives the solution

$$\theta(\tau) = \theta_0 + \epsilon\tau + \epsilon^{\frac{\sigma}{2-\sigma}} F_2(k, \Omega) g(\theta_0) \sin(\tau). \quad (\text{C.2})$$

The correction to the zeroth-iteration solution is of the order of $\mathcal{O}(\epsilon^{\frac{\sigma}{2-\sigma}})$, and recall that we are interested here in parameters for which $0 < \sigma/(2-\sigma) < 1$. The linear term, that is $\mathcal{O}(\epsilon)$, is therefore subdominant; however, it is kept in the solution since the oscillatory term has a vanishing contribution to the integral (2.6) for $A(I, \theta)$.

Iterating this solution back to the equations of motion, and expanding the function $g(\theta)$ around $\theta = \theta_0$, the equation of motion reads

$$\begin{aligned} \frac{d\theta}{d\tau} = & \epsilon + \epsilon^{\frac{\sigma}{2-\sigma}} F_2(k, \Omega) g(\theta_0) \cos(\tau) + \epsilon^{\frac{\sigma}{2-\sigma}+1} F_2(k, \Omega) g'(\theta_0) \tau \cos(\tau) \\ & + \epsilon^{2\frac{\sigma}{2-\sigma}} F_2(k, \Omega)^2 g(\theta_0) g'(\theta_0) \sin \tau \cos \tau. \end{aligned} \quad (\text{C.3})$$

The correction is $\mathcal{O}(\epsilon^{2\frac{\sigma}{2-\sigma}})$, or $\mathcal{O}(\epsilon^{\frac{\sigma}{2-\sigma}+1})$, depending on the value of σ . In any case it is self-consistent to take the first iterate (C.2) as the leading-order solution. Inserting this solution into the integral defining $A(I, \theta)$, (2.6), where now the time dependence of θ includes also the oscillatory term, the integrand can be expanded around the linear part $\theta_0 + \epsilon\tau$:

$$\begin{aligned} A(I, \theta) = & \frac{k}{\Omega} \int_0^{2\pi} V(I) g(\theta_0 + \epsilon\tau) \cos \tau \, d\tau \\ & + \frac{k}{\Omega} \int_0^{2\pi} V(I) g'(\theta_0 + \epsilon\tau) \epsilon^{\frac{\sigma}{2-\sigma}} F_2(k, \Omega) g(\theta_0) \cos \tau \sin \tau \, d\tau. \end{aligned} \quad (\text{C.4})$$

Expanding g' around θ_0 , it is seen that the oscillatory term gives a vanishing contribution to the integral, with a correction which is of higher order in ϵ . One should therefore consider whether higher-order terms in (C.3) give a non-vanishing contribution which cannot be neglected compared to the term linear in ϵ . Inspection of the form of the solution, (C.3), reveals that those terms which cannot always be neglected compared to the linear term, for example the terms which is $\mathcal{O}(\epsilon^{2\frac{\sigma}{2-\sigma}})$, give a contribution to the integral of A which is a higher order in ϵ (they vanish if the linear part of the argument of g' is neglected). It is easily checked that this is also true for the next iterations, and therefore the leading order in ϵ is consistently given by (2.7).

The dimensionless equations of motion for the case of $\sigma < 0$ are

$$\frac{d\epsilon}{d\tau} = \epsilon^{\frac{\sigma-4}{\sigma-2}} F_3(k, \Omega) \left(\frac{dg}{d\theta} \right) \cos(\tau) \quad (\text{C.5})$$

$$\frac{d\theta}{d\tau} = \epsilon + \epsilon^{\frac{2}{2-\sigma}} F_4(k, \Omega) g(\theta) \cos(\tau). \quad (\text{C.6})$$

Also in this case, the time variation of ϵ is slower than that of θ , and the equation of motion for θ can be solved first with ϵ taken to be constant. Because of considerations similar to those in the case discussed above, the oscillatory term is of lower order in ϵ and an iterative procedure can be self-consistently employed to give the leading order of the solution. Equation (2.7) is therefore a leading order in ϵ also for negative values of σ .

Appendix D. Matrix elements for the $\sigma < 0$

In this appendix we calculate the semiclassical matrix elements of the position and momentum variables for the model discussed in section 3. We use again the simplest semiclassical estimate, namely the Fourier components of the functions $x(I, \theta)$ and $p(I, \theta)$. It will be shown that the type of singularity of the potential at the origin (related to the value of σ), determines in a simple way the asymptotic behaviour of these matrix elements.

The unperturbed Hamiltonian is

$$\mathcal{H}_0(I) = \frac{1}{\alpha} I^\alpha \tag{D.1}$$

where α is defined in (3.3). In this case $-\infty < \alpha < 0$, and the energies are negative. The function $\theta(x, I)$ is derived from the generating function $S(x, I)$, and for $x \ll x_0$ (where x_0 is the turning point) it is approximated as follows:

$$\theta(x, I) = \frac{\partial S}{\partial I} = \frac{\partial E}{\partial I} \int^x \frac{m}{\sqrt{2m(E + by^\sigma)}} dy \sim \omega(I) \int^x y^{-\sigma/2} dy. \tag{D.2}$$

Therefore, ignoring constants, we have $\theta(x, I) \propto I^{\alpha-1} x^{1-\sigma/2}$, or

$$x(I, \theta) \propto I^\gamma \theta^{\frac{2}{2-\sigma}} \tag{D.3}$$

where $\gamma = 2/(2 + \sigma)$.

For the special case of the hydrogen atom, ($\sigma = -1$), $x \propto I^2 \theta^{2/3}$ which agrees with the exact results for $x \ll x_0$. The exact function $x(I, \theta)$ is shown in figure A3 for various values of σ . The position variable has a cusp at $\theta = 0$, where the function behaves as θ^ρ and its derivative diverges ($\rho < 1$). This point dominates the asymptotic behaviour of the Fourier components, which decay asymptotically as $1/|m|^{\rho+1}$ [44]. Applying it to the position variable in our model, it is found that the Fourier components decay with a power

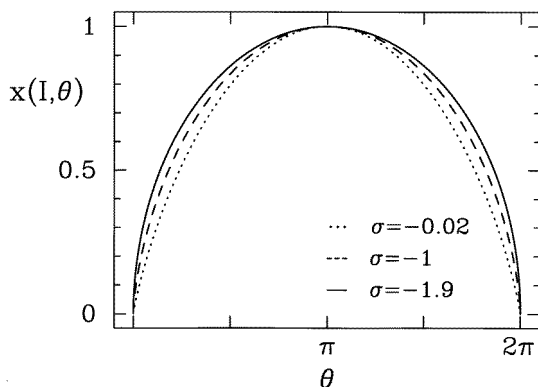


Figure A3. Same as figure 6, for negative values of σ .

$\eta = (4 - \sigma)/(2 - \sigma)$ as a function of m . The semiclassical dipole-matrix elements therefore behave as

$$\langle n|\hat{x}|m\rangle \propto \frac{(n\hbar)^\eta}{|n-m|^\eta}. \quad (\text{D.4})$$

For the case of the hydrogen atom, $\sigma = -1$, one finds that

$$\langle n|\hat{x}|m\rangle \propto \frac{(n\hbar)^2}{|n-m|^{5/3}} \quad (\text{D.5})$$

in agreement with the asymptotic behaviour of the exact matrix elements. Similar properties can now easily be found for $p(I, \theta)$. At the origin $x \approx 0$,

$$p(I, \theta) = \sqrt{2m(E(I) + bx^\sigma)} \approx \sqrt{2mb}x^{\sigma/2} \propto I^{\frac{\sigma}{\sigma+2}}\theta^{\frac{\sigma}{2-\sigma}}. \quad (\text{D.6})$$

Because the potential is $(-\infty)$ at the origin, the momentum p always diverges there. The variable $p(I, \theta)$ is antisymmetric with respect to $\theta = \pi$, therefore the Fourier components are pure imaginary. Thus the expansion coefficients of the θ dependent part of p satisfy

$$g_m \sim iC_2 \operatorname{sgn}(n-m)/|m|^{\frac{2}{2-\sigma}} \quad (\text{D.7})$$

and the matrix elements of the momentum semiclassically are

$$\langle n|\hat{p}|m\rangle \sim iC_2 \operatorname{sgn}(n-m) \frac{(n\hbar)^{\frac{\sigma}{2+\sigma}}}{|n-m|^{\frac{2}{2-\sigma}}}. \quad (\text{D.8})$$

For the hydrogen atom, this implies $|\langle n|\hat{p}|m\rangle| \propto \frac{n^{-1}}{|n-m|^{2/3}}$ in agreement with the exact matrix elements.

The next term in the expansion around $x = 0$ in (D.2), gives the correction to the momentum variable (D.6) as

$$p(I, \theta) \sim I^{\frac{\sigma}{\sigma+2}} (\bar{a}\theta^{\frac{\sigma}{2-\sigma}} + \bar{b}\theta^{\frac{-\sigma}{2-\sigma}}) \quad (\text{D.9})$$

where \bar{a} and \bar{b} are constants. Similar to the case of positive σ , this also shows that in the next order in the expansion the product form of $p(I, \theta)$ is preserved. It leads to a correction to the Fourier components (D.7) $g_m \sim C_2/m^{2/2-\sigma} + C'_2/m^{2(1-\sigma)/2-\sigma}$, which renormalizes the function $A(I, \theta)$ by a factor of $(1 + (C'_2/C_2)\epsilon^{-2\sigma/2-\sigma})$. Thus, the conclusions of section 3 remain qualitatively unchanged to the next order in ϵ .

Appendix E. Stationary-phase calculation of $A(I; \theta)$ for the hydrogen atom

In this appendix, we derive the approximate expression (4.3) for the function $A(I; \theta)$ for the case of the hydrogen atom, in a different way. The reason for doing this is that the approximation suggested in appendix A, although reasonable and confirmed by numerical calculations, is not a controlled one. For the hydrogen atom exact expressions are known for the transformation to action-angle variables; this enables us to calculate the leading order in the asymptotic expansion of the integral (2.7), where ϵ is the small parameter.

The Hamiltonian is (4.1) with $\sigma = -1$. The transformation to action-angle variables is

$$x(I, \theta) = 2I^2 \sin^2(\psi/2) \quad (\text{E.1})$$

$$p(I, \theta) = \frac{1}{I} \cot(\psi/2) \quad (\text{E.2})$$

$$\theta = \psi - \sin \psi. \quad (\text{E.3})$$

To first order in the field strength k , one may use the approximation (2.5) for the Floquet matrix elements. The perturbation is now expressed by coupling to the momentum operator, therefore

$$A(\theta) = \frac{k}{\Omega} \int_0^T p(I; \theta + \omega t) \cos \Omega t \, dt \tag{E.4}$$

$$= \frac{kI^2}{\Omega} \int_{\psi_1}^{\psi_2} \cos[(\psi - \sin \psi - \theta)/\epsilon] \sin \psi \, d\psi \tag{E.5}$$

where the initial angle θ is constant and is related to the endpoints of the integral by $\psi_1 - \sin \psi_1 = \theta$, and $\psi_2 - \sin \psi_2 = \theta + 2\pi\epsilon$. The second expression holds, consistently, to first order in k [25, 24]. It is convenient to consider the following integral

$$\mathcal{I} = \int_{\psi_1}^{\psi_2} \exp[i(\psi - \sin \psi - \theta)/\epsilon] \sin \psi \, d\psi. \tag{E.6}$$

The leading order to this integral comes from the stationary point $\psi = 2\pi$, if it is contained inside the integration domain. The contribution of this point has been calculated by Casati *et al* [25]. Taking only this part into account, one finds that

$$\mathcal{I} \cong \begin{cases} 0 & 0 < \theta < 2\pi(1 - \epsilon) \\ \frac{2\pi}{i} (2\epsilon)^{2/3} \text{Ai}'(0) e^{i(2\pi - \theta)/\epsilon} & 2\pi(1 - \epsilon) < \theta < 2\pi \end{cases} \tag{E.7}$$

implying for $A(\theta)$ the following form:

$$A(\theta) \cong \begin{cases} 0 & \theta < 2\pi(1 - \epsilon) \\ R \sin[(\theta - 2\pi)/\epsilon] & \theta > 2\pi(1 - \epsilon) \end{cases} \tag{E.8}$$

where $R \cong 0.82\pi k/\Omega^{5/3}$ is a constant, independent of I (or ϵ). The minus sign relative to (4.3) is because for the hydrogen atom $C_2 < 0$.

The region of parameters where this approximation is zero, is that where the stationary point lies outside the interval of integration. The next-order contribution to \mathcal{I} comes from the endpoints. Expanding the phase around the endpoint ψ_1 into the complex plane, $\psi = \psi_1 + i\zeta$, $0 \leq \zeta \leq \infty$, one finds the correction from the endpoints to be $E(\psi_1) - E(\psi_2)$, where

$$E(\psi) = \frac{1}{2} \frac{e^{i\psi} [(1 - \cos \psi)/\epsilon - 1] - e^{-i\psi} [(1 - \cos \psi)/\epsilon + 1]}{(1 - \cos \psi)^2/\epsilon^2 - 1}. \tag{E.9}$$

This correction holds if the endpoints of the interval are not too close to the stationary point. For typical values of ψ_1 and ψ_2 , it can easily be seen that the real part of the endpoints correction is of the order of ϵ^2 , whereas the imaginary part is of the order of ϵ . The resulting contribution to $A(\theta)$ is of the order of $\epsilon^{4/3}$ which is small compared with (E.8).

Appendix F. A ‘delta like’ property of the ‘sinc’ function

In this appendix it is shown that for any function $\mathcal{F}(M)$ which can be expanded in a Laurent series, the following equation holds:

$$\sum_{M=-\infty}^{\infty} \mathcal{F}(M) \text{sinc}[\pi(M + \delta)] = \mathcal{F}(-\delta). \tag{F.1}$$

Consider first the case of an expansion with positive powers, a Taylor expansion:

$$\mathcal{F}(M) = \sum_{k=1}^{\infty} \frac{\mathcal{F}^{(k)}(0)}{k!} M^k. \tag{F.2}$$

Changing the order of summation, one needs to evaluate infinite sums as in (F.1) with powers M^k . It is convenient to use the integral representation

$$\operatorname{sinc}[\pi(M + \delta)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(M+\delta)y} dy. \quad (\text{F.3})$$

Then the sum may be written as

$$\sum_{M=-\infty}^{\infty} \operatorname{sinc}[\pi(M + \delta)] M^k = (-i)^k \frac{d^k}{dx^k} \left\{ \sum_M \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(M+\delta)y} dy e^{iMx} \right\}_{x=0}. \quad (\text{F.4})$$

The sum over M gives a periodic delta function, $\sum_j \delta(x + y - 2\pi j)$. Since x is an arbitrarily small variable and y is in the interval $[-\pi, \pi]$, only one delta function remains, corresponding to $j = 0$. Integrating dy one finds

$$\sum_{M=-\infty}^{\infty} \operatorname{sinc}[\pi(M + \delta)] M^k = (-i)^k \frac{d^k}{dx^k} \{e^{-i\delta x}\}_{x=0} = (-\delta)^k. \quad (\text{F.5})$$

Therefore (F.1) follows for a function which has a Taylor expansion. Similarly for negative powers, using integration instead of differentiation with respect to x , one finds that (F.1) holds.

References

- [1] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
- [2] Blümel R, Fishman S and Smilansky U 1986 *J. Chem. Phys.* **84** 2604
- [3] Casati G, Chirikov B V, Shepelyansky D L and Guarneri I 1987 *Phys. Rep.* **154** 77
- [4] Giannoni M J, Voros A and Zinn-Justin J (ed) 1991 *Chaos and Quantum Physics (Proc. of the Les-Houches Summer School, Session LII, 1989)* (Amsterdam: North-Holland)
- [5] Casati D, Guarneri I and Smilansky U (ed) 1993 *Quantum Chaos (Proc. of the International School of Physics 'Enrico Fermi', Varenna, July 1991)* (New York: North-Holland)
- [6] Haake F 1991 *Quantum Chaos* (New York: Springer)
- [7] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
- [8] Casati G, Chirikov B V, Izrailev F M and Ford J 1979 *Stochastic Behaviour in Classical and Quantum Hamiltonian Systems (Lecture Notes in Physics 93)* ed G Casati and J Ford (Berlin: Springer)
- [9] Fishman S, Grepel D R and Prange R E 1982 *Phys. Rev. Lett.* **49** 509
Grepel D R, Prange R E and Fishman S 1984 *Phys. Rev. A* **29** 1639
- [10] Moore F L, Robinson J C, Bharucha C, Williams P E and Raizen M G 1994 *Phys. Rev. Lett.* **73** 2974
- [11] Moore F L, Robinson J C, Bharucha C, Sundaram B and Raizen M G 1996 *Phys. Rev. Lett.* **75** 4598
- [12] Fishman S 1993 *Quantum Chaos (Proc. of the International School of Physics 'Enrico Fermi', Varenna, July 1991)* ed D Casati, I Guarneri and U Smilansky (New York: North-Holland)
- [13] Prange R E and Nee T W 1968 *Phys. Rev.* **168** 779
- [14] Shimshoni E and Smilansky U 1987 *Nonlinearity* **1** 435
- [15] Benvenuto F, Casati G, Guarneri I and Shepelyansky D L 1991 *Z. Phys. B* **24** 159
- [16] de Oliveira C R, Guarneri I and Casati G 1994 *Europhys. Lett.* **27** 187
- [17] Shepelyansky D L 1993 *Quantum Chaos (Proc. of the International School of Physics 'Enrico Fermi', Varenna, July 1991)* ed D Casati, I Guarneri and U Smilansky (New York: North-Holland)
- [18] Blümel R, Hillermeier C and Smilansky U 1990 *Z. Phys. D* **15** 267
- [19] Blümel R, Buchleitner A, Graham R, Sirko L, Smilansky U and Walther H 1991 *Phys. Rev. A* **44** 4521
- [20] Leopold J G and Richards D 1988 *J. Phys. B: At. Mol. Opt. Phys.* **21** 2179
- [21] Brenner N and Fishman S 1995 *J. Phys. A: Math. Gen.* **28** 5973
- [22] Zeldovich Y B 1967 *Sov. Phys.-JETP* **24** 1006
- [23] Casati G and Molinari L 1989 *Prog. Theor. Phys. Suppl.* **98** 287
- [24] Leopold J G and Richards D 1989 *J. Phys. B: At. Mol. Opt. Phys.* **22** 1931
- [25] Casati G, Guarneri I and Shepelyansky D L 1988 *IEEE J. Quantum Electron.* **24** 1240
- [26] Cocks S and Reichl L E *Phys. Rev. A* **53** in press
- [27] Levit S and Smilansky U 1977 *Ann. Phys.* **103** 198; 1977 *Ann. Phys.* **108** 165

- [28] Richards D, Leopold J G and Jensen R V 1989 *J. Phys. B: At. Mol. Opt. Phys.* **22** 417
- [29] Percival I C and Richards D 1970 *J. Phys. B: At. Mol. Phys.* **3** 1035
- [30] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (Oxford: Pergamon)
- [31] Gontis V and Kaulakys B 1987 *J. Phys. B: At. Mol. Phys.* **20** 5051
- [32] Grepel D R, Fishman S and Prange R E 1982 *Phys. Rev. Lett.* **49** 833
- [33] Prange R E, Grepel D R and Fishman S 1984 *Phys. Rev. B* **29** 6500
- [34] Berry M V 1983 *Physica* **10D** 369
- [35] Jensen R V, Leopold J G and Richards D R 1988 *J. Phys. B: At. Mol. Opt. Phys.* **21** L527
- [36] Chirikov B V 1979 *Phys. Rep.* **52** 263
- [37] Griniasty M and Fishman S 1988 *Phys. Rev. Lett.* **60** 1334.
- [38] Brenner N and Fishman S 1992 *Nonlinearity* **4** 211
- [39] Shirley J H 1965 *Phys. Rev. B* **138** 979
- [40] Howland J S 1974 *Math. Ann.* **207** 315
- [41] Bellissard J 1985 *Trends and Developments in the Eighties* ed S Albeverio and P Blanchard (Singapore: World Scientific)
- [42] Delyon F, Simon B and Soulliard B 1984 *Phys. Rev. Lett.* **52** 2187; 1985 *Ann. Inst. H. Poincaré* **42** 283
- [43] Buchleitner A, Delande D, Zakrzewski J, Mntegna R N, Arndt M and Walther H 1995 *Phys. Rev. Lett.* **75** 3818
- [44] Lighthill M J 1980 *An Introduction to Fourier Analysis and Generalized Functions* (Cambridge: Cambridge University Press)